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Permutable polarities and a class of ovoids of the Hermitian surface

A. Cossidente[a,*], G.L. Ebert[b]

[a] *Dipartimento di Matematica, Università della Basilicata, Contrada Macchia Romana, 85100 Potenza, Italy*
[b] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*

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Abstract

Using some geometry of quadrics permutable with a Hermitian surface $\mathcal{H}(3, q^2)$ of $PG(3, q^2)$, q odd, a class of ovoids of $\mathcal{H}(3, q^2)$ admitting the group $PGL_2(q)$ is constructed.

1. Introduction

A non-degenerate Hermitian variety, defined as the set of all self-conjugate points of a non-degenerate unitary polarity of $PG(r, q^2)$, provides an important example of a finite classical polar space. The combinatorial properties of non-degenerate Hermitian varieties have been studied together with those of quadrics and linear complexes in the general theory of polar spaces (see [4, 5]). In this setting important objects are the ovoids. An *ovoid* $\mathcal{O}$ of a non-degenerate Hermitian variety $\mathcal{H}(r, q^2)$, $r \geq 3$, is a set of points of $\mathcal{H}(r, q^2)$ which has exactly one common point with every generator of $\mathcal{H}(r, q^2)$. Here *generator* means a maximal totally singular projective subspace of $\mathcal{H}(r, q^2)$. In even dimensions r, Thas [13] proved that $\mathcal{H}(r, q^2)$ has no ovoid. In odd dimensions r, the existence problem is still open for $r > 3$, apart from some special cases settled with a negative answer by Blokhuis and Moorhouse (see [9, 15]).

In this paper we are interested in ovoids of $\mathcal{H}(3, q^2)$. Lines lying on $\mathcal{H}(3, q^2)$ are the generators of $\mathcal{H}(3, q^2)$, and the size of an ovoid is q^3+1. The intersection of the Hermitian surface $\mathcal{H}(3, q^2)$ with any of its non-tangent planes is a Hermitian curve $\mathcal{H}(2, q^2)$, which

* Corresponding author.
E-mail addresses: cossidente@unibas.it (A. Cossidente), ebert@math.udel.edu (G.L. Ebert).

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is easily seen to be an ovoid (called the *classical ovoid*). The following construction due to Payne and Thas [10] provides non-classical ovoids of $\mathcal{H}(3, q^2)$ for every q. Given a classical ovoid $\mathcal{O}$ of $\mathcal{H}(3, q^2)$, choose two distinct points P_1 and P_2 on $\mathcal{O}$. Then the line ℓ through P_1 and P_2 meets $\mathcal{O}$ at $q + 1$ points. Replace these points with those in the intersection of $\mathcal{H}(3, q^2)$ with the polar line of ℓ. The resulting set contains no conjugate pairs of points and has the same size as $\mathcal{O}$. Hence it is an ovoid. More generally, by starting from any ovoid $\mathcal{O}$ of $\mathcal{H}(3, q^2)$, one can consider the hyperbolic line L spanned by two points P_1 and P_2 of $\mathcal{O}$. Then, if all isotropic points of L lie on $\mathcal{O}$, one can show that $(\mathcal{O} \cup L^{\perp})\backslash L$ is an ovoid of $\mathcal{H}(3, q^2)$. Indeed, suppose that a point X of $L^{\perp}$ is conjugate to some point Y of $\mathcal{O}\backslash L$. Then XY is one of the $q+1$ lines of $\mathcal{H}(3, q^2)$ through X and hence contains some point Z of L. But Z and Y are both points of $\mathcal{O}$, a contradiction. Hence, no point of $L^{\perp}$ is conjugate to a point of $(\mathcal{O} \cup L^{\perp})\backslash L$, and $(\mathcal{O} \cup L^{\perp})\backslash L$ is an ovoid in this case. We will call such a procedure *derivation*.

There are so many mutually inequivalent ovoids of $\mathcal{H}(3, q^2)$ known that their classification seems to be possible only under some extra condition(s) (see Section 4). For instance, in [2] all transitive ovoids of $\mathcal{H}(3, q^2)$, q even, have been classified.

Here we are concerned with the construction of a class of ovoids of $\mathcal{H}(3, q^2)$, q odd, which admit the linear group $PGL_2(q)$ in their automorphism group. Our construction mainly relies on the theory of quadrics that are permutable with a Hermitian surface $\mathcal{H}(3, q^2)$ of $PG(3, q^2)$ (see [11]). Throughout this paper $q = p^h$ will denote an odd prime power.

2. Preliminaries

Let $\mathcal{A}$ be an orthogonal polarity commuting with the Hermitian polarity $\mathcal{U}$ associated with $\mathcal{H}(3, q^2)$. Set $\mathcal{V} = \mathcal{A}\mathcal{U} = \mathcal{U}\mathcal{A}$. As Segre pointed out [11, p. 136], $\mathcal{V}$ fixes either $(q + 1)^2$ or $q^2 + 1$ points on $\mathcal{H}(3, q^2)$, yielding a hyperbolic quadric $Q^+(3, q)$ or an elliptic quadric $Q^-(3, q)$, respectively, embedded in a Baer subgeometry B of $PG(3, q^2)$. In both cases $B \cap \mathcal{H}(3, q^2) = Q^{\pm}(3, q)$. Notice that the points of $\mathcal{H}(3, q^2)$ fixed under $\mathcal{V}$ are those admitting the same tangent plane with respect to both the orthogonal polarity and the unitary polarity.

Hence, the projective orthogonal group $PGO_4^{\epsilon}(q)$, $\epsilon = \pm$, associated with $\mathcal{V}$ is seen to be a subgroup of the projective unitary group $PGU_4(q^2)$ associated with $\mathcal{U}$. In terms of forms, let us assume that (V, g) is a 4-dimensional unitary space over $GF(q^2)$. Let F be the subfield of $GF(q^2)$ of index two. Choose a basis $\mathbf{b} = \{v_1, \ldots, v_4\}$ of V such that $g(v_i, v_j) \in F$ for all i and j, and let W denote the F-span of $\mathbf{b}$. The restriction $\bar{g}$ of g to W is a non-degenerate symmetric bilinear form. If $\mathbf{b}$ is an orthonormal basis, then the discriminant of $\bar{g}$ is a square. Replacing v_1 by βv_1, where β is a generator of $GF(q^2)^*$, the discriminant of $\bar{g}$ becomes a non-square. Therefore, we obtain embeddings $O_4^{\epsilon}(q) < GU_4(q^2)$ for both $\epsilon = +$ and $\epsilon = -$. For more details on these group embeddings, see [6].

3. The ovoid construction

We start this section with the following technical lemma.

Lemma 3.1. *Let $\mathcal{H}(2, q^2)$ be a Hermitian curve of $PG(2, q^2)$. Let B be a Baer subplane of $PG(2, q^2)$ intersecting $\mathcal{H}(2, q^2)$ in a conic C of B. Then the stabilizer G of C in $PGU_3(q^2)$ has three orbits on $\mathcal{H}(2, q^2)$.*

Proof. It is sufficient to show that G has two orbits on the points of $\mathcal{H}(2, q^2)\backslash C$. Given such a point P, by [12, Theorem 2] $P^\perp \cap B$ is a point Q. The two orbits correspond to the following two cases: Q is external to C and Q is internal to C. There are $q(q+1)/2$ external points to C and $q(q-1)/2$ internal points to C in B. By Witt's theorem, the isometry group of C is transitive both on external points and on internal points, and these isometries extend to $\mathcal{H}(2, q^2)$ via the embedding $PO_3(q) \leq PGU_3(q^2)$. Hence, it is sufficient to show in each case that the stabilizer of Q in $PO_3(q)$ is transitive on the points of $\mathcal{H}(2, q^2)\backslash C$ conjugate to Q. If Q is external to C, then $|Q^\perp \cap C| = 2$. Denoting by $\bar{Q}^\perp$ the extension of $Q^\perp$ to $GF(q^2)$, we have $|\bar{Q}^\perp \cap \mathcal{H}(2, q^2)| = q + 1$ and the cyclic group of order $q - 1$ contained in the stabilizer of Q in $PO_3(q)$ is regular on points of $(\bar{Q}^\perp \cap \mathcal{H}(2, q^2))\backslash C$. Similarly, if Q is an internal point to C, then $|Q^\perp \cap C| = 0$ and the cyclic group of order $q + 1$ contained in the stabilizer of Q in $PO_3(q)$ is regular on the $q + 1$ points of $(\bar{Q}^\perp \cap \mathcal{H}(2, q^2))\backslash C$, proving the result. Note that each of the two orbits distinct from C has size $q(q^2 - 1)/2$. $\square$

Let $\mathcal{H} = \mathcal{H}(3, q^2)$ be the Hermitian surface of $PG(3, q^2)$, q odd, with equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$, where $X_0, \ldots, X_3$ are homogeneous coordinates in $PG(3, q^2)$. Let $\{Q_a \mid a \in GF(q^2)^* := GF(q^2)\backslash\{0\},\ a^{q+1} = 1\}$ denote a family of $q + 1$ quadrics of $PG(3, q^2)$, where Q_a has equation $aX_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$. Straightforward computations show that each of these quadrics is hyperbolic and any two of them intersect in the conic $\bar{C}$, given by equation $X_1^2 + X_2^2 + X_3^2 = 0$, lying in the plane $\bar{\pi}$ with equation $X_0 = 0$. Let π denote the Baer subplane of $\bar{\pi}$ whose normalized point coordinates lie in the subfield $F = GF(q)$, and let $C = \bar{C} \cap \pi$ denote the associated subconic of $\bar{C}$ in π. Furthermore, let $\mathcal{U} = \mathcal{H} \cap \bar{\pi} \cong \mathcal{H}(2, q^2)$ be the Hermitian curve, given by equation $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$, that one obtains by intersecting the Hermitian surface $\mathcal{H}$ with the plane $\bar{\pi}$.

Lemma 3.2. *Using the above notation, $C = \mathcal{H} \cap \pi = \mathcal{U} \cap \bar{C} = \mathcal{H} \cap \bar{C}$.*

Proof. Since $x^{q+1} = x^2$ for $x \in F$, it suffices to show that $\mathcal{U} \cap \bar{C} \subseteq C$. Let $P = (0, x_1, x_2, x_3) \in \mathcal{U} \cap \bar{C}$. Then $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0 = x_1^2 + x_2^2 + x_3^2$. If $x_1 = 0$, then without loss of generality we may assume $x_2 = 1$ and $x_3^{q+1} = -1 = x_3^2$, implying that $x_3 \in F$ and $P \in \pi \cap \mathcal{U} = C$. If $x_1 \neq 0$, then we may assume $x_1 = 1$ and $x_2^{q+1} + x_3^{q+1} = -1 = x_2^2 + x_3^2$. In particular, $x_2^{2q} + x_3^{2q} = (-1)^q = -1,\ x_3^{2q} = -(1 + x_2^{2q})$ and $x_3^2 = -(1 + x_2^2)$. Thus $x_3^{2q+2} = (1 + x_2^2)(1 + x_2^{2q})$ on the one hand, and $x_3^{2q+2} = (1 + x_2^{q+1})^2$ on the other. This implies that $(x_2 - x_2^q)^2 = 0$ and $x_2 \in F$. This further implies $x_3 \in F$ from the equation $x_2^{q+1} + x_3^{q+1} = x_2^2 + x_3^2$, and hence $P \in C$ as before. $\square$

From [11, p. 146] each quadric Q_a is permutable with $\mathcal{H}(3, q^2)$. In particular, $(q+1)/2$ of them, say $Q_{a_1}, \ldots, Q_{a_{(q+1)/2}}$, are such that $\mathcal{H}(3, q^2) \cap Q_{a_i}$ is an elliptic quadric $\mathcal{O}_i$ embedded in a Baer subgeometry B_i of $PG(3, q^2)$, for $i = 1, 2, 3, \ldots, (q+1)/2$.

Also, from [11, Section 75] $B_i \cap \mathcal{H}(3, q^2) = \mathcal{O}_i$ for each i. Here, a_i is such that $a_i^{(q+1)/2} = -1$, and hence by appropriate renumbering we may assume that $a_i = \beta^{(q-1)(2i-1)}$ for $i = 1, 2, \ldots, (q+1)/2$, where β is a primitive element of $GF(q^2)$.

In fact, one can explicitly describe the above Baer subgeometries B_i. With a_i defined as above, let $\eta_i = \beta^{-(2i-1)}$ for $i = 1, 2, \ldots, (q+1)/2$, and consider the Baer subgeometry $B_i = \{(x_0, x_1\eta_i, x_2\eta_i, x_3\eta_i) : x_0, x_1, x_2, x_3 \in F\}$. To show that B_i is the Baer subgeometry described above, it suffices to show that $\mathcal{H} \cap Q_{a_i} \subseteq B_i$.

Lemma 3.3. *Using the above notation, we have* $\mathcal{H} \cap Q_{a_i} \subseteq B_i$ *for* $i = 1, 2, \ldots, (q+1)/2$.

Proof. We use the ordered basis $\{(1, 0, 0, 0), (0, \eta_i, 0, 0), (0, 0, \eta_i, 0), (0, 0, 0, \eta_i)\}$, where i is temporarily fixed, to represent points of $PG(3, q^2)$. Let $P = (y_0, y_1, y_2, y_3) \in \mathcal{H} \cap Q_{a_i}$, where the homogeneous coordinates are with respect to the above basis. If $y_0 = 0$, the argument is similar to the proof of Lemma 3.2 and is left as an exercise. Hence we assume $y_0 \neq 0$, and thus $y_0 = 1$ without loss of generality. Therefore the equations $1 + \eta_i^{q+1}(y_1^{q+1} + y_2^{q+1} + y_3^{q+1}) = 0 = a_i + \eta_i^2(y_1^2 + y_2^2 + y_3^2)$ hold. To simplify notation, we let $\lambda = \omega^{2i-1}$, where $\omega := \beta^{q+1}$ is a primitive element of the subfield F. Note that λ is a nonsquare in F. Direct computation shows that

$$\frac{a_i}{\eta_i^2} = \lambda = \frac{1}{\eta_i^{q+1}}.$$

Hence we have the two equations $y_1^{q+1} + y_2^{q+1} + y_3^{q+1} + \lambda = 0$ and $y_1^2 + y_2^2 + y_3^2 + \lambda = 0$.

From the second equation and its qth power, we obtain $\frac{y_1^2}{\lambda} + \frac{y_2^2}{\lambda} = -(1 + \frac{y_3^2}{\lambda})$ and $\frac{y_1^{2q}}{\lambda} + \frac{y_2^{2q}}{\lambda} = -(1 + \frac{y_3^{2q}}{\lambda})$. Multiplying these two equations and clearing denominators yields

$$y_1^{2q+2} + y_2^{2q+2} - y_3^{2q+2} = \lambda^2 + \lambda y_3^2 + \lambda y_3^{2q} - y_2^2 y_1^{2q} - y_1^2 y_2^{2q}.$$

On the other hand, from the first equation above we have $(y_3^{q+1} + \lambda)^2 = (y_1^{q+1} + y_2^{q+1})^2$, which upon simplification yields

$$y_1^{2q+2} + y_2^{2q+q} - y_3^{2q+q} = \lambda^2 + 2\lambda y_3^{q+1} - 2y_1^{q+1} y_2^{q+1}.$$

Comparing the latter two equations yields $\lambda(y_3 - y_3^q)^2 = (y_2 y_1^q - y_1 y_2^q)^2$, implying that $y_3,\ y_2 y_1^q \in F$ since λ is a nonsquare in F.

From the original pair of equations we now have $y_1^{q+1} + y_2^{q+1} = y_1^2 + y_2^2 \in F$. An argument analogous to that given in the proof of Lemma 3.2 then shows that $y_2 \in F$ and hence $y_1 \in F$, implying that $P \in B_i$ and completing the proof. □

It should be remarked that using the above coordinates one can directly (and easily, without the use of [11]) show that $Q_{a_i} \cap \mathcal{H} = Q_{a_i} \cap B_i = \mathcal{H} \cap B_i$ is an elliptic quadric $\mathcal{O}_i$ defined over F. In so doing, the fact that λ is a nonsquare in F is used once again.

Note that the coordinate description of the Baer subgeometries implies that for $i \neq j$, $B_i \cap B_j = \pi \cup \{(1, 0, 0, 0)\}$. It then follows from Lemma 3.2 that $\mathcal{O}_i \cap \mathcal{O}_j = C$ for $i \neq j$.

Proposition 3.4. *The union $\mathcal{E} = \bigcup \mathcal{O}_i$, $i = 1, \ldots, (q + 1)/2$, is a partial ovoid of $\mathcal{H}$ of size $(q^3 + q + 2)/2$.*

Proof. Since each quadric Q_{a_i} is permutable with $\mathcal{H}$, each elliptic quadric $\mathcal{O}_i$ is a partial ovoid of $\mathcal{H}$. Let $\mathcal{O}_i, \mathcal{O}_j$ be two distinct elliptic quadrics of $\mathcal{E}$. Let P be a point of $\mathcal{O}_i \backslash \mathcal{O}_j$. Then by [12, Theorem 2] $P^\perp$ (with respect to the unitary polarity) meets the Baer subgeometry B_j in a line ℓ. Since Q_{a_i} is permutable with $\mathcal{H}$, $P^\perp$ coincides with the tangent plane to the elliptic quadric $\mathcal{O}_i$ at P. Furthermore, since $\mathcal{O}_i$ is an ovoid of B_i, $P^\perp$ meets the plane of the conic C, namely π, in a line that must be external to C. This means that ℓ is an external line to $\mathcal{O}_j$, and $\mathcal{O}_i \cup \mathcal{O}_j$ is a partial ovoid of $\mathcal{H}$. From this it follows that $\mathcal{E}$ is a partial ovoid of $\mathcal{H}$. Since $\mathcal{O}_i \cap \mathcal{O}_j = C$ for $i \neq j$, $|\mathcal{E}| = \frac{1}{2}(q + 1)(q^2 - q) + (q + 1) = \frac{1}{2}(q^3 + q + 2)$. $\square$

It should be noticed that using the above coordinates, one can give a direct proof of this proposition.

The next step in our construction is to adjoin to the partial ovoid $\mathcal{E}$ one of the two orbits of G on the Hermitian curve $\mathcal{U} = \mathcal{H} \cap \bar{\pi}$, where G is the stabilizer of C in $PGU_3(q^2)$ as defined in Lemma 3.1.

Theorem 3.5. *The union of $\mathcal{E}$ and the G-orbit $\mathcal{B}$ on $\mathcal{U} \cong \mathcal{H}(2, q^2)$ corresponding to external points of C is an ovoid $\mathcal{O}$ of $\mathcal{H} = \mathcal{H}(3, q^2)$.*

Proof. Let P be a point of $\mathcal{E}$. Then $P \in \mathcal{O}_i$, for some i with $1 \leq i \leq (q + 1)/2$. The plane $P^\perp$ (with respect to the unitary polarity) meets $\bar{\pi}$ in a line L. Since $P^\perp$ coincides with the tangent plane to $\mathcal{O}_i$ at P, as observed above, the line L meets B_i in a line r. Since $\mathcal{O}_i$ is a partial ovoid, r cannot be secant or tangent to C, and so r is an external line to C. Thus $r^\perp$ (with respect to the orthogonal polarity) is an internal point of C, and hence P is not conjugate to any point of $\mathcal{B}$. Since $\mathcal{E}$ and $\mathcal{B}$ are both partial ovoids, so is $\mathcal{O} = \mathcal{E} \cup \mathcal{B}$. But $\mathcal{O}_i \cap \bar{\pi} = C$ for all i, and hence $\mathcal{E} \cap \mathcal{B} = \emptyset$. Therefore $|\mathcal{O}| = (q^3 + q + 2)/2 + (q^3 - 2)/2 = q^3 + 1$ and $\mathcal{O}$ is an ovoid of $\mathcal{H}$. $\square$

Alternative Proof. It suffices to show that no two points of $\mathcal{O}$ determine a generator of $\mathcal{H}$. To this end, suppose $P \in \mathcal{B}$ and $R \in \mathcal{E}$ such that $m = PR$ is a generator of $\mathcal{H}$. Then $P \in \mathcal{U}$ lies on some secant line ℓ of C. Let $\ell \cap C = \{A_1, A_2\}$. Also $R \in \mathcal{O}_i \backslash C$ for some i. Thus $\Gamma = \langle \ell, m \rangle$ is a plane defining a Baer subplane $\Gamma_0 = \Gamma \cap B_i$ of B_i. Now Γ_0 meets $\mathcal{O}_i$ in a conic D_0 defined over the field F. Moreover, Γ meets the hyperbolic quadric Q_{a_i} in a conic D which contains D_0 as a subconic. Hence $D \cap \mathcal{H} = \Gamma \cap Q_{a_i} \cap \mathcal{H} = \Gamma \cap \mathcal{O}_i = \Gamma \cap B_i \cap Q_{a_i} = D \cap B_i = D_0$. Recall that Γ is a tangent plane to $\mathcal{H}$ at some point $V \in m$. Since A, B and P are distinct collinear points of $\mathcal{H} \cap \Gamma$ (lying on the line ℓ) and since $V \notin \ell$, the distinct lines VA_1, VA_2 and $VP = m$ are necessarily generators of $\mathcal{H}$. But $\mathcal{O}_i$ is a partial ovoid of $\mathcal{H}$, and hence none of these generators can meet $\mathcal{O}_i$ at more than one point. As $D_0 \subset \mathcal{O}_i$ and $D_0 = D \cap \mathcal{H}$, each of these three generators must be tangent to the conic D in the plane Γ. Hence we have three concurrent tangents to a conic in a plane of odd characteristic, a contradiction. This proves the result. $\square$

Proposition 3.6. *The automorphism group* $G = \mathrm{Aut}(\mathcal{O})$ *of* $\mathcal{O}$ *is a subgroup of the stabilizer of the point* $P = (1, 0, 0, 0)$ *in* $PGU_4(q^2)$*. In particular,* $G/K \simeq PGL_2(q)$*, where* K *is the cyclic homology group of order* $(q^2 - 1)$ *with center* P *and axis* $\bar{\pi}$*.*

Proof. Clearly, G is a subgroup of the stabilizer of P in $PGU_4(q^2)$, the stabilizer being isomorphic to $GU_3(q^2)$. Let K be the cyclic homology group with center P and axis $\bar{\pi}$. Then $GU_3(q^2)/K$ is isomorphic to $PGU_3(q^2)$. By construction K fixes $\mathcal{O}$ and thus $G/K < PGU_3(q^2)$. By construction again G stabilizes the conic C. From [8], the stabilizer of C in $PGU_3(q^2)$, which is isomorphic to $PGL_2(q)$, is maximal in $PGU_3(q^2)$. Now, consider the induced action of G on the plane $\bar{\pi}$. The kernel of this action is exactly K. Since $G/K < PGU_3(q^2)$ and G stabilizes C, [8] implies that $G/K \simeq PGL_2(q)$. $\square$

Proposition 3.7. *The ovoid* $\mathcal{O}$ *constructed above can be obtained from multiple derivation from the classical ovoid.*

Proof. From the previous proposition $\mathrm{Aut}(\mathcal{O})$ contains the cyclic homology group of order $q^2 - 1K$ with center P and axis $\bar{\pi}$. The result now follows from [3]. $\square$

We call the family of ovoids of $\mathcal{H}(3, q^2)$ constructed above *permutable ovoids*.

4. Spreads of $Q^-(5, q)$ and 1-Systems of $Q(6, q)$

Let $\mathcal{L}$ denote the set of lines of $PG(3, q^2)$. In the Grassmannian mapping $\Phi : \mathcal{L} \to Q^+(5, q^2)$, by which lines of $\mathcal{L}$ are mapped to points of $Q^+(5, q^2)$, the incidence structure consisting of all points and lines of $\mathcal{H}(3, q^2)$ is isomorphic to the dual of the incidence structure consisting of all points and lines of $Q^-(5, q)$. This also proves a classical isomorphism of the corresponding collineations groups. Under such an isomorphism, ovoids $\mathcal{O}$ of $\mathcal{H}(3, q^2)$ and 1-spreads of $Q^-(5, q)$ are equivalent objects. We recall that a 1-spread of $Q^-(5, q)$ is a partition of the point set of $Q^-(5, q)$ into lines. Let L be a fixed line of some 1-spread $\mathcal{S}$ of $Q^-(5, q)$. For each line M of $\mathcal{S}$, the subspace $\langle L, M\rangle$ has dimension three and intersects $Q^-(5, q)$ in a non-singular hyperbolic quadric $Q^+(3, q)$. Let $\mathcal{R}_{L,M}$ be the regulus of $Q^+(3, q)$ containing L and M. Then, $\mathcal{S}$ is *locally Hermitian* with respect to L if $\mathcal{R}_{L,M}$ is contained in $\mathcal{S}$ for all lines M of $\mathcal{S}$ different from L. If $\mathcal{S}$ is locally Hermitian with respect to all the lines of $\mathcal{S}$, then $\mathcal{S}$ is called *Hermitian* (see [14]). In [1] the Hermitian spread of the generalized hexagon $H(q)$ has been characterized as a spread of $Q^-(5, q)$ which is locally Hermitian with respect to at least two of its lines. In [7], it has been observed that this property also holds for Hermitian spreads of $Q^-(5, q)$ and therefore, we can characterize a Hermitian spread of $Q^-(5, q)$ as a spread, which is locally Hermitian with respect to at least two, and hence all, of its lines. In particular, $\mathcal{S}$ is Hermitian if and only if the corresponding ovoid $\mathcal{O}$ of $\mathcal{H}(3, q^2)$ is classical.

We have the following theorem.

Theorem 4.1. *The* 1*-spread* $\mathcal{S} = \mathcal{O}^{\Phi}$ *of* $Q^-(5, q)$*, where* $\mathcal{O}$ *is a permutable ovoid constructed as in Theorem* 3.5*, admits the group* $PGL_2(q)$ *and it is not locally Hermitian.*

Proof. The first part of the statement is clear. Let H_i be the stabilizer of $\mathcal{O}_i$ in $PGU_4(q^2)$, $i = 1, \ldots, (q+1)/2$. Then $\mathrm{Stab}_{H_i}(C)$ acts transitively on $\mathcal{O}_i \backslash C$ and since one can find an element of $\mathrm{Stab}_{PGU_4(q^2)}(\mathcal{O})$ which maps $\mathcal{O}_i$ onto $\mathcal{O}_j$, it follows, using Lemma 3.1, that $\mathrm{Aut}(\mathcal{O})$ has three orbits on $\mathcal{O}$, namely C, $\mathcal{E}\backslash C$ and $\mathcal{B}$. Suppose that $\mathcal{S}$ is locally Hermitian with respect to the line L. Let P denote the point of $\mathcal{O}$ corresponding by duality to L. We distinguish three cases depending upon the orbit to which P belongs. Assume first that $P \in C$. Let ℓ be any one of the many hyperbolic lines through P consisting entirely of points of $\mathcal{O}$ that meets $C \cup \mathcal{B}$ only in P. Then $|\ell \cap (\mathcal{E}\backslash C)| = q$. From [11, p. 136], if ℓ meets an elliptic quadric $\mathcal{O}_i$ in a further point, then $|\ell \cap B_i| = q + 1$; namely, $\ell \cap B_i$ is a line of B_i. Since there are $(q-1)/2$ elliptic quadrics in $\mathcal{E}$ distinct from $\mathcal{O}_i$, there exists j, with $1 \le j \le (q+1)/2$, such that $|\ell \cap \mathcal{O}_j| \ge 3$, a contradiction. Assume next that $P \in \mathcal{E}\backslash C$, and so $P \in \mathcal{O}_i$ for some i, $1 \le i \le (q+1)/2$. In this case, there exists an hyperbolic line ℓ through P consisting entirely of points of $\mathcal{O}$ such that $|\ell \cap (C \cup \mathcal{B})| = 1$. Put $Q = \ell \cap (C \cup \mathcal{B})$. If $Q \in C$, then since ℓ cannot meet $\mathcal{O}_i$ in any further points as above, it follows that $|((\bigcup_{j \neq i} \mathcal{O}_j)\backslash C) \cap \ell| = q - 1$, and again an elliptic quadric contained in $\mathcal{E}$ would contain three collinear points. On the other hand, if $Q \in \mathcal{B}$, then ℓ cannot intersect π in any further points. Arguing as above, ℓ intersects an elliptic quadric $\mathcal{O}_i$ of $\mathcal{E}$ in at least two points, and hence $B_i \cap \ell$ is a line of B_i. The line $B_i \cap \ell$ intersects the Baer subplane containing C in a point and so ℓ lies in π, a contradiction. Finally, assume that $P \in \mathcal{B}$. Then there exists a hyperbolic line through P consisting entirely of points of $\mathcal{O}$ that does not meet π in any further points, and we fall into the previous case. We conclude that $\mathcal{S}$ cannot be locally Hermitian. $\square$

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